Título do trabalho:

Estimation of Correlations in Credit Risk in Brazil

AUTORES
RICARDO GONÇALVES SILVA
Serasa S.A.
ricardosilva@serasa.com.br

FABIO WENDLING MUNIZ DE ANDRADE
Pontifícia Universidade Católica de São Paulo
fandrade@serasa.com.br

Resumo

Correlações no risco de crédito constituem um tópico bastante importante em finanças. Sob o novo acordo de Basiléia, as correlações constituem um parâmetro de vital importância para a avaliação de risco de um portfólio e para o cálculo do capital econômico dos bancos. Neste artigo são propostas duas formas alternativas para a estimação das correlações: modelagem bayesiana hierárquica e ajuste de distribuições. Para este objetivo, foram utilizados dados de mais de 300.000 empresas brasileiras, classificadas em cinco diferentes categoria de risco. Além disso, foram gerados através de simulações de Monte Carlo, carteiras hipótesticas, de diferentes tamanhos amostrais e com correlações distintas. A estimativa do parâmetro de correlação obtidas nessas simulações é interpretado como o tamanho do viés de pequena amostra das carterias. Os principais resultados, tanto na abordagem Bayesiana como no ajuste de distribuiçõe, mostram que o valor estimado da correlação entre os ativos das empresas brasileiras é bastante pequeno se comparado ao valor proposto no acordo de Basiléia, impactando diretamente no montante de capital regulatório requerido dos bancos pelo acordo de Basiléia II.

Palavras-chave:

Bassel II, default correlation, estimação Bayesiana, simulação de Monte Carlo, Ajuste de Distribuições.

Abstract

A leading topic in empirical finance is correlation among risk assets. Under the new Basel II Accord, correlations are regarded as parameters extremely important for credit risk evaluations on a portfolio basis and for banks' economic capital requirements. In this paper we propose two different approaches to estimate correlations: hierarchical Bayes modeling and distribution fit. Main results show that the asset's correlation value for the Brazilian firm's is remarkably less than the value proposed by the Basel Accord. We also detected that distribution fit methods are biased, so, the estimated parameters' coefficients should be corrected. To do so we conduct some Monte Carlo experimentation and compute the bias' size.

Keywords:

Bassel II, default correlation, Bayesian estimations, Monte Carlo simulation, Distribution Fit.

Estimation of Correlations in Credit Risk in Brazil

1. Introduction

Credit risk plays a major role in portfolio management, derivatives pricing and banking capital regulation. All the financial instruments with contingent payments have an associated probability that an issuer will default on its obligations in the future, thus, giving rise to a credit risk problem.

Default correlation is a measure of the dependence among risks. Along with default rates and recovery rates, it is a necessary input in the estimation of the portfolio credit risk. In general, the concept of default correlation incorporates the fact that systemic events cause the default event to cluster. Coincident movements in default among borrowers may be triggered by common, underlying factors. Within the context of retail portfolios, systemic events might include macroeconomic events such as changes in the rate of unemployment or geographically specific events. Default correlation is defined by Nagpal e Bahar (2001) as the relationship between default probabilities and joint default probabilities. They note that historical rates of default support the idea that credit events are correlated. This correlation is a critical factor in the estimation of the tails of the overall credit loss distributions. Thus, failure to recognize the impact of shocks to the portfolio through default correlation will ultimately underestimate the measures of risk and economic capital required to manage that risk.

There are several methodologies currently employed in the development of correlations within portfolios as discussed in Zhou (1997). For example, Loffler (2003) estimates default correlations based on the joint distribution of asset values. As discussed in Crouhy, Galai e Mark (2000), equity prices are often used as a proxy to estimate asset correlations given that asset values are not directly observable. One commonly employed method is the identification of a benchmark for the purpose of developing asset return correlations and then mapping these into default correlations. The approach requires making assumptions about the relationship between asset prices and default. However, this approach is not applicable within a retail context as there is no asset price for the individual borrower. Alternatively, correlations can be inferred from historical default volatilities as described generally within Appendix F of the *CreditMetrics* Technical Document (1997).

According to structural credit risk models, as in Merton (1974), Duffie and Singleton (1999), and Eom, Helvege and Huang (2004), among many others, the correlation between default occurrences arises from the relation between company assets values. In the literature concerning estimations of correlations, a variety of values has been found, ranging from 0.5% to 50%. Gordy e Heitfield (2002), using S&P and Moody's data found values ranging on the interval (5.1%; 13.5%), using maximum likelihood estimation. Again, Demey et al. (2004), found 9.4% of intra-sector correlation. Using the same methodology, Rösch (2003), using Germany data, found general correlation of 0.86% and, clustered by economic sectors, values from 0.54% to 3.52%. Finally, Göessl (2005), using S&P data and bayesian techniques trough Markov Chain Monte Carlo (MCMC), found a general value of 9.41%.

In this paper we estimate correlations using a one factor structural model as first presented by Vasicek (1991). The objectives of the paper are threefold. First we apply a bayesian hierarchical approach, using diffuse prior specifications, to estimate the default correlation parameters. Second, we conduct the same analysis using a classic model and compare the results. In light of these results, we concluded that the classical results are biased towards portfolios' small sample. Then, finally, we estimate the bias's size by means of Monte Carlo Simulation.

The paper is organized as follows. Next section presents a simple credit portfolio model; section 3 provides the bayesian hierarchical technique used and the obtained results. Section 4 introduces the distribution fit approach, and the results and the bias's corrected estimates obtained from Monte Carlo Simulation. Finally, section 5 concludes the paper.

2. A Credit Portfolio Model

Consider the one factor model, as introduced by Vasicek (1991). Following Göessl (2005), let us consider a portfolio of N credit risky assets a_i, $i = 1,\ldots,N$, each comprising one unit. Each of these assets can be classified into one of some finite risk classes $r_i \in \{R_1,\ldots,R_K\}$, defining its risk profile completely. The rating classes have individual one year probabilities of default $\mathbf{h}_{R_j}$, $j = 1,\ldots K$, where the probability of default of a single debtor (PD), $\mathbf{h}_i = \mathbf{h}_{R_j}$, for $r_i = R_j$.

Further, let us suppose the respective instruments follow a one factor asset value model, i.e., it is assumed that a firm defaults when its asset value process $Z_{i,t}$ falls below the firm's liabilities or a certain default frontier. The corresponding default frontier or threshold is determined by its individual risk profile or risk class r_i. Let now $Z_{i,t}$ be described by a logarithm Wiener process:

$$dZ_i = \mu_i Z_i dt + \sigma_i A_i dz_i \tag{1}$$

Given that:

$$E(dz_i)^2 = dt$$
$$E\left[(dz_i)(dz_j)\right] = \rho dt, i \neq j$$

a discrete version of (1) on a t year horizon is defined as follows:

$$\begin{gathered} Z_{i,t} = F_t\sqrt{\rho} + U_{i,t}\sqrt{1-\rho}, i = 1,\ldots,N, \\ \rho \in [0,1], F_t \sim \mathbf{N}(0,1), U_{i,t} \sim \mathbf{N}(0,1)\, iid. \end{gathered} \tag{2}$$

where F_t constitute the company exposure to common factor, invariant throughout the portfolio and $U_{i,t}$ an idiosyncratic component, resembling an independent individual contribution of asset i to its evolution over time. These two components are weighted by a correlation parameter, ρ, determining the intra-portfolio dependencies within the whole portfolio[1].

[1]This model also corresponds to Gordy e Heitfield (2003)'s restrictions R1 and R3.

In order to obtain the probability of a given portfolio loss some specific exposure to common factor, Vasicek uses Merton's (1974) diffusion model and, assuming an infinite number of exposures of equal amounts in the portfolio and equi-correlation among the asset value of the borrowing companies, shows that the cumulative default rate distribution at default rate x is given by:

$$F(X) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\left(\sqrt{1-\rho}\right)\Phi^{-1}(x) - \Phi^{-1}(\mathbf{h}_{R_j})\right)\right) \tag{3}$$

where Φ and Φ^{-1} are the cumulative standard normal distribution function and its inverse function respectively. We shall use equation (3) is section 4, in order to adjust the distribution fit model.

3. The Bayesian and MCMC methods

Let's first briefly review the main facts concerning bayesian statistics. Let z be a vector containing some data we wish to analyze. The basic idea behind the bayesian approach is based on the Bayes theorem:

$$p(\theta \mid z) = \frac{p(z,\theta)}{p(z)} = \frac{p(z \mid \theta)p(\theta)}{\int p(z \mid \theta)d\theta} \tag{4}$$

which relates the observed data to some unknown parameters. Here, the main inferential difference from the classical procedures is the fact that ρ is no longer supposed to be fixed but having a probability distribution, which also can be specified. For this task we define a prior distribution which combined with the information about the parameters contained in the data sample give us the conditional posterior parameters distribution. This resulting posterior distribution yields all information regarding the unknown parameters.

For the formulation of prior distributions there are two general cases to be discerned. The first is the case where the prior probability measures the degree of belief that an individual has in an uncertain proposition, and is in that respect subjective. In this case there is a natural choice for the prior distributions. The second and more common case arises when there is no previous knowledge about the parameters to be estimated. Box e Tiao (1973) define a non-informative prior as one that provides little information relative to the experiment. For this purpose usually overdispersed or flat prior are applied, e.g., uniform distributions on the unit interval or flat normal distributions for metric unrestricted parameters with a sufficient large variance to allow for a practically overdispersed distribution on a reasonable interval for the variance parameter.

The major drawback in Bayesian inference lies on the fact that, despite some simple and tractable models, the computations involved in solving the integral in the denominator of equation (2) are usually analytically intractable. Even thought the principles of algorithms which were able to overcome this limitation were already presented by Metropolis et al. (1953) and Hastings (1970), only the dramatic increase in computational power finally allowed the application in statistics. Once started by publications of e.g. Besag, York e Mollie (1991), Smith e Roberts (1993) or Gilks, Richardson e Spiegelhalter (1996), the so-called Markov chains Monte Carlo (MCMC) techniques became the method of choice in Bayesian Statistics given their flexibility, robustness, and almost unlimited applicability.

In order to state the Bayesian model, we note that the probability of asset i defaults is given by $P(X_i < k_i) = \mathbf{h}_i \Rightarrow k_i = \Phi^{-1}(\mathbf{h}_i)$. Further, conditional on the portfolio factor F_t this equation becomes $P(X_i < k_i \mid F_t = f_t) = \mathbf{h}_{i|f_t}$. Then,

$$\mathbf{h}_{i|f_t} = \Phi\left(\frac{\Phi^{-1}(\mathbf{h}_i) - \sqrt{\rho} f_t}{\sqrt{1-\rho}}\right) \tag{5}$$

completing a generalized linear model for Bernoulli data with a probit link function.

According to Göessl (2005), considering $\mathbf{Bin}(n, \mathbf{h}, k)$ the Binomial distribution, without loss of generality and for simplicity at this point it is supposed that in case of default every obligor suffers the same loss of one unit, i.e. their loss given default is 100%. Defining $\mathbf{L_t} = (L_{t1}, \ldots, L_{tK})$, $\mathbf{n_t} = (n_{t1}, \ldots, n_{tK})$, $\mathbf{h_t} = (h_{R1}, \ldots, h_{RK})'$, and also considering the other unknown parameters, the complete conditional likelihood function for period t and a one year horizon can be rewritten in a concise form:

$$P(\mathbf{L}_t = \mathbf{l}_t \mid \mathbf{n}_t, \mathbf{h}, f_t, \rho) = \prod_{j=1}^{K} \mathbf{Bin}(n_{t,R_j}, h_{j|f_t}, l_{t,R_j}) \tag{6}$$

Finally, assuming the case of no substantial prior information and therefore flat priors for the parameters, we complete our hierarchical Bayesian credit portfolio approach. After some algebraic manipulation, the corresponding posterior distribution of the unknown parameters conditional on the observations $(\mathbf{n}, \mathbf{l})$ is given by:

$$P(\mathbf{h}, \rho, \mathbf{f} \mid \mathbf{n}, \mathbf{l}) = \frac{\prod_{j=1}^{T} P(\mathbf{L}_t = \mathbf{l}_t \mid \mathbf{n}_t, \mathbf{h}, f_t, \rho) p(\mathbf{h}) p(\rho) p(\mathbf{f})}{P(\mathbf{L} = \mathbf{l} \mid \mathbf{n})}, \tag{7}$$

where:

$$h_{R_j} \sim \mathbf{U}(0,1),\ i.i.d.,\ j = 1, \ldots, K,$$
$$\rho \sim \mathbf{U}(0,1)$$
$$f_t \sim \mathbf{N}(0,1),\ i.i.d.,\ j = 1, \ldots, K,$$

Thereby defining $\mathbf{U}(0,1)$ as the uniform distribution in the unit interval and $\mathbf{f} = (f_1, \ldots, f_T)'$, $P(\mathbf{L} = \mathbf{l} \mid \mathbf{n})$ can be calculated by integrating the numerator of equation (6) with respect to $\mathbf{h}$, ρ, and $\mathbf{f}$.

The above model is easily estimated using the WinBUGS package. Basically, considering $\mathbf{n_t}$ debtors in the portfolio, and $\mathbf{l_t}$ the number of defaults for given k risks classes in time t ,we specify a binomial distribution ($\mathrm{Bin}(\mathbf{n}_{\mathrm{kt}}, \mathbf{l}_{kt})$) for the number of defaults. Further, with a *uniform prior* for the correlation parameter, $\rho \sim \mathrm{U}(0,1)$, and a *normal* assumption for $F_t \sim \mathbf{N}(0,1)$, for the common factor, we are able to estimate the benchmark distribution given by equation (5) by means of Markov chain Monte Carlo techniques trough the Gibbs sampling algorithm (an excellent review of Gibbs sampling algorithm can be found in Geman and Geman (1984)).

The empirical data used in this work was supplied by Serasa S.A., the major Brazilian Credit Bureau, and includes monthly risk classification and default information for 367.500 Brazilian companies from January 1998 to July 2006. The default's concept used here is based on the idea of payment delays over then 90 days. We distinguish five samples based on information of firm's net incoming and asset's value. The rule for classification, which establishes the companies' size classification used by Serasa S.A. in its credit reports, is showed Table 1 below:

Table 1: Serasa S.A. Rules for risk classification

If Net Sales	or Assets	Classification	Sample Size
<= R$ 1.2 million	<= R$ 1.2 million	Size 1	228.654
<= R$ 4.0 million	<= R$ 4.0 million	Size 2	31.473
<= R$ 25 million	<= R$ 25 million	Size 3	23.093
<= R$ 50 million	<= R$ 50 million	Size 4	2.458
> R$ 50 million	> R$ 50 million	Size 5	5.933

Using this data, the aim is to derive the joint posterior distribution (JPD) of the portfolio correlation coefficient. We generated JPD's samples using Markov chains of a length of 30.000, taking every 3rd state of the chains, with a burn-in phase of 10.000 iterations. The number of iterations was monitored according the guidelines presented by Rosenthal (1994). Table 1 below shows the main results, in percentages.

Table 2: Correlation results from the Bayesian approach.

Firm's Size	Mean	Std. Dev.	2.50%	97.50%	PSR Factor	DIC
Size 1	2.53%	0.39%	1.88%	3.40%	~ 1	3103.71
Size 2	0.97%	0.16%	0.70%	1.31%	~ 1	3284.78
Size 3	1.93%	0.34%	1.38%	2.66%	~ 1	2155.2
Size 4	3.53%	0.96%	1.92%	5.66%	~ 1	1158.5
Size 5	2.55%	1.06%	0.78%	4.91%	~ 1	891.995

As expected, the posterior densities for the estimated parameters, showed in Figure 1, conforms to the assumption of normality, since the combination of a normal likelihood with an uniform prior leads to a normal distribution (see Bernardo and Smith (1994), for details).

Convergence was stated based on the Gelman and Rubin (1992)'s potential scale reduction factor (PSR) and the Spiegelhalter, Best and Linde (2002) deviance information criteria (DIC)[2]. Figure 2 below shows the trace graphics for each segment.

[2] The potential scale reduction factor was close to unity for all models, indicating convergence.

Figure 1: Posterior Densities for estimated correlation parameter, ρ.

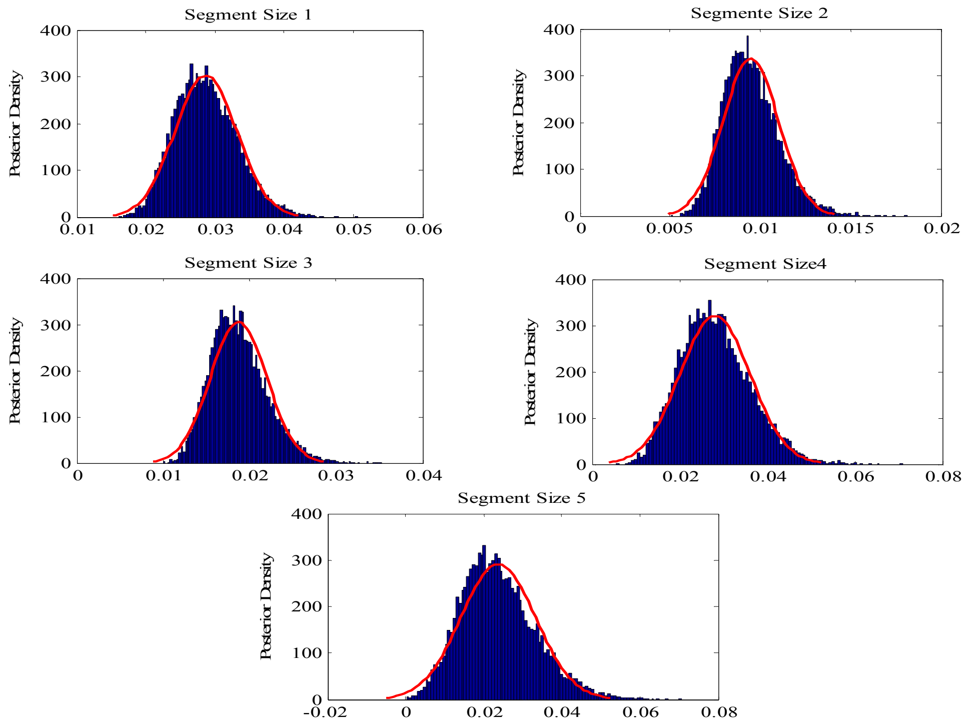
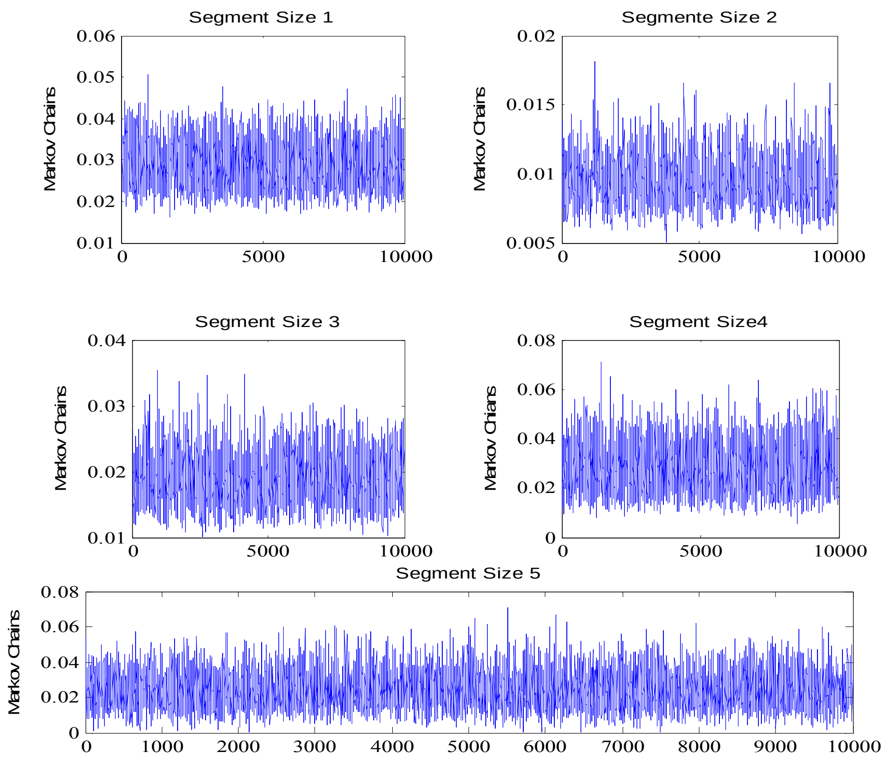


Figure 2: Markov chains realizations for 10.000 iterations.



According to Basel II the correlation parameter is set to 15% for residential mortgages exposures and 4% for revolving exposures. For other retail credit exposures the correlation has to be calculated as a weighted average of two extreme values by the following equation:

$$\rho = \rho_{\min}\left(\frac{1-\exp(b * PD)}{1-\exp(-b)}\right) + \rho_{\max}\left(\frac{1-\exp(b * PD)}{1-\exp(-b)}\right) \tag{8}$$

where: b = 35, ρ_{min} = 0.03 and ρ_{max} = 0.16.

Then, the first conclusion we found indicates the asset's correlation value for the Brazilian firm's is remarkably less than the value proposed by the Basel Accord. This results potentially impacts on the capital adequacy rules, setting down the amount of capital a bank or credit institution must hold. Let's now see the results obtained from the distribution fit approach.

4. Distribution fit model and results

In order to compare and also validate the results we obtained in applying the Bayesian methodology, we also consider a classical distribution fit approach. The idea behind this model is straightforward: since Vasicek (1991) presented the closed form for the asset's distribution in a portfolio, the aim is to apply a procedure which implements iterative methods that attempt to find estimates for non-linear models, specifically for the correlation parameter of the default's rate distribution.

Technically speaking, nonlinear estimation is a general fitting procedure that will estimate any kind of relationship between a dependent (or response variable), and a list of independent variables. In general, all regression models may be stated as:

$$y_i = f(\mathbf{X}, \boldsymbol{\rho}) + \varepsilon_i \tag{9}$$

in which the functional part of the model is not linear with respect to the unknown parameters, ρ, and we are able specify either standard least squares estimation or maximum likelihood estimation. For advanced details of the distribution fitting approach, see Karian and Dudewicz (2000).

In the case of our credit portfolio model, the idea is consider the formulae observed in equation (3):

$$F(X) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\left(\sqrt{1-\rho}\right)\Phi^{-1}(x) - \Phi^{-1}(\mathbf{h}_{R_j})\right)\right) \tag{10}$$

and, using a nonlinear model estimation procedure, computes estimates for ρ. For this task we choose to use the method of non-linear least squares to estimate the values of the unknown parameters, trough the Gauss-Newton optimization algorithm. Results are presented in table (3) below:

Table 3: Correlation results from the Distribution Fit approach.

Firm's Size	Mean	Std. Dev.	2.50%	97.50%	F Value	Pr > F
Size 1	0.28%	0.01%	0.27%	0.29%	107607	<.0001
Size 2	0.33%	0.01%	0.31%	0.34%	56852.8	<.0001
Size 3	0.81%	0.03%	0.75%	0.86%	29373.1	<.0001
Size 4	3.84%	0.07%	3.71%	3.98%	70351.4	<.0001
Size 5	3.93%	0.15%	3.64%	4.22%	18579.1	<.0001

Again, our results show estimations for the asset's correlation lower then the values proposed by the Basel II Accord. Usually, default's rate variations are due to two sources: stochastic variations (*Small Portfolio Effect*) and correlations (*Systematic Factor Effect*). However, the Vasicek's model used here, fairly based on a cumulated distribution function, which has the correlation ρ as it parameter, and assuming an infinite portfolio, doesn't take into account the first source, leading to biased estimation of correlations when equation (10) is used in a distribution fit procedure.

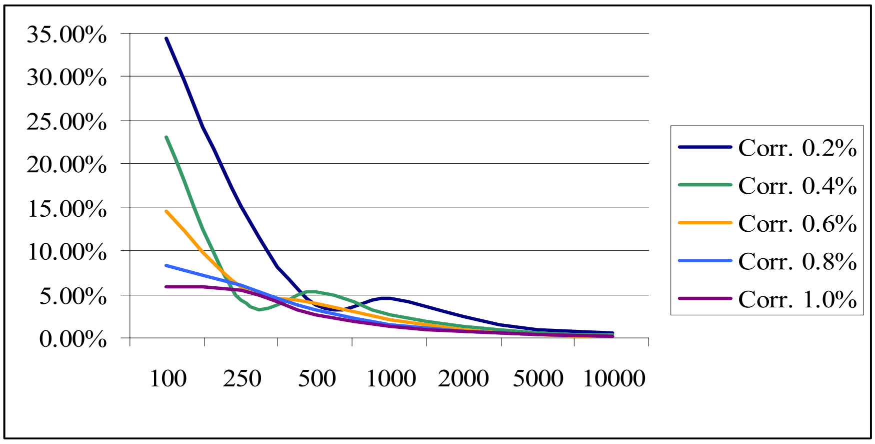
To evaluate the bias's size we conducted a Monte Carlo experiment for different default rates and portfolio's sizes. We simulate default's in hypothetical portfolios where all debtors have the same probability of default and defaults are supposed to be independent. Using the data from these hypothetical portfolios built under the assumption of non-correlated companies, we estimate the correlation parameter ρ using the distribution fit approach. In fact, in this case, the estimated ρ correlation coefficient is a measure of bias in correlation estimation of portfolios of same size and default rate. Table 4 below present the results obtained from 25.000 draws:

Table 4: Monte Carlo simulations of estimates' bias.

Portfolio's Size	Monthly Default Rate				
	0,2%	**0,4%**	**0,6%**	**0,8%**	**1,0%**
100	34,5%	23,0%	14,6%	8,4%	5,8%
250	15,2%	4,4%	5,9%	6,0%	5,5%
500	3,8%	5,3%	3,9%	3,2%	2,7%
1000	4,6%	2,7%	2,0%	1,6%	1,4%
2000	2,4%	1,4%	1,0%	0,8%	0,7%
5000	1,0%	0,6%	0,4%	0,3%	0,3%
10000	0,5%	0,3%	0,2%	0,2%	0,1%

As expected, the Monte Carlo simulation shows that the bias's size vanishes with as bigger as the portfolio's size. The graph 1 below is quite intuitive:

Graph 1: Evolution of estimation's bias as function of portfolio's sizes.



Although ρ estimates from distribution fit applied to non-infinite portfolios have a bias, it could be argued that, for the purpose of getting an accurate default distribution, it is better to use the biased parameter then the unbiased one when using the Vasicek model, once real portfolios always have a finite number of companies.

5. Concluding Comments

In light of the main empirical results, two conclusions are straightforward: first, the asset's correlation value for the Brazilian firm's is remarkably less than the value proposed by the Basel Accord. Clearly these results reflect the structural characteristics of the firms in Brazilian, and couldn't, naturally, be extending to other countries.

However, it's important to note that many empirical findings, as the ones in Gordy e Heitfield (2002), Demey et al. (2004), Rösch (2003), and Göessl (2005), among others has pointed to the direction of correlations values out and away bellow the ones recognized by Bassel II Accord.

Our findings indicate that the regulatory capital required by the Bassel II formulae could be as closer as possible from the real economic capital needed by the portfolio if the estimated values of the assets' correlation parameter, ρ, were used of instead the ones imposed by the Basel II Accord.

References

BERNARDO, J. M.; SMITH, A. F. M. *Bayesian Theory*. New York: Wiley, 1994**.**

BESAG, J.; YORK, J.; MOLLIE, A. Bayesian image restoration with two applications in spatial statistics. *Annals of the Institute of Statistical Mathematics*, v. 1, n. 43, p. 1–59, 1991.

BLACK, F.; COX J. C. Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance*, n. 31, p. 351-367, 1976.

BOX, G. E. P.; TIAO, G. *Bayesian Inference in Statistical Analysis*. New York: Wiley, 1973.

CROUHY, M.; GALAI, D.; MARK, R. A comparative analysis of current credit risk models. *Journal of Banking and Finance*, n. 24, p. 59–117, 2000.

DEMEY, P., JOUANIN, J., ROGET, C. and RONCALI, T. Maximum likelihood estimate of default correlation. *Credit Risk*, n. 6, p. 104–108, 2004.

DUFFIE, D.; SINGLETON, K.J. Modeling term structures of defaultable bonds, *Review of Financial Studies*, n. 12, v. 4, p. 687-720.

GELMAN, A. A.; RUBIN, D. B. Inference from iterative simulation using multiple sequences. *Statistical Science*, n. 7, p. 457–511, 1992.

GEMAN S. and D. GEMAN. Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, v. 6, p. 721-741, 1984.

GILKS, W. R.; RICHARDSON, S.; SPIEGELHALTER, D. J. *Markov Chain Monte Carlo in Practice*. London: Chapman & Hall, 1996.

GöESSL, C. *Predictions based on certain uncertainties: a Bayesian credit portfolio approach*. [S.l.], 2005.

GORDY, M.; HEITFIELD, E. *Estimating default correlations from short panels of credit rating performance data*. [S.l.], 2003.

HATINGS, W. K. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, n. 57, p. 97–109, 1970.

KARIAN, Z. A.; DUDEWICZ E. J. *Fitting statistical distributions*. Boca Raton, Florida: CRC Press, 2000.

LOFFLER, G. The effects of estimation error on measures of portfolio credit risk. *Journal of Banking and Finance*, v. 27, p. 1427–1453, 2003.

MERTON, R. C. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, n. 29, p. 449–470, 1974.

METROPOLIS, N. N. et al. Equations of state calculations by fast computing machines. *Journal of Chemical Physics*, n. 21, p. 1087–1091, 1953.

NAGPAL, K.; BAHAR, R. Measuring default correlation. *Risk*, n. 14, p. 129–132, 2001.

ROSENTHAL, J. Theoretical rates of convergence for Markov chain Monte Carlo., 1994. Technical Report, University of Toronto, Canada.

RÖSH, D. (2003). Correlations and business cycles of credit risk: evidence from bankruptcies in Germany, *Financial Markets and Portfolio Management,* n. 17, p. 309-331.

SMITH, A. F.; ROBERTS, G. O. Bayesian computation via the Gibbs sampler and related Markov chain Monte Carlo methods (with discussion). *Journal of the Royal Statistical Society, Series B,* n. 55, p. 3–23, 1993.

SPIEGELHALTER N. G. BEST, B. P. C. D. J.; LINDE, A. V. der. Bayesian measures of model complexity and fit (with discussion). *Journal of the Royal Statistical Society, Series B,* n. 64, p. 583–616, 200

VASICEK, O. *The loan loss distribution*. [S.l.], 1991.

EOM, Y. H.; HELWEGE, J.; HUANG J.Z. Structural Models of Corporate Bond Pricing: An Empirical Analysis, *Review of Financial Studies*, n. 17, v. 2, p. 499 – 544, 2004.