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Minimum Variance Fuzzy Possibilistic Portfolio

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ABSTRACT

Portfolio selection using mean-variance models within a probabilistic framework assumes, essentially, that the situation of financial markets in future can be reflected by security data in the past. Because of the information incompleteness and the complexity of financial market, it is impossible to precisely predict the future return and the actual risk of a portfolio, since this environment is affected by non-probabilistic factors such that the return risky asset is fuzzy uncertainty. This paper evaluates the performance of minimum variance fuzzy possibilistic portfolio in the Brazilian equity market and compares its results to those of the following benchmarks: IBOVESPA equity index, minimum variance portfolio from crisp (real) numbers, an equally-weighted portfolio, and with the maximum Sharpe ratio portfolio. The numerical results show that the minimum variance fuzzy possibilistic portfolio has higher returns with lower risk compared to all remaining benchmarks, being easily replicable by investors, institutions and all market participants in general.

KEYWORDS: portfolio selection; fuzzy possibilistic theory; minimum variance portfolio.

RESUMO

Seleção de carteiras de acordo com o princípio de média-variância em uma abordagem probabilística assume, essencialmente, que os movimentos futuros dos mercados podem ser derivados a partir de dados do passado. Como as informações são incompletas e o mercado financeiro apresenta comportamento complexo, a predição do retorno futuro e do risco incorrido de uma carteira de investimento mostra-se desafiadora, uma vez que fatores não probabilísticos afetam tal ambiente, de forma que os retornos de ativos com risco são caracterizados por incertezas, em termos nebulosos. Este artigo avalia o desempenho de uma carteira nebulosa possibilística de variância mínima no mercado de ações brasileiro e compara seus resultados com os seguintes *benchmarks*: índice IBOVESPA, carteira de variância mínima considerando dados reais, uma carteira igualmente ponderada, e uma carteira obtida pela maximização do índice de Sharpe. Os resultados mostraram que a carteira nebulosa possibilística de variância mínima produziu os maiores retornos com um baixo nível de risco, quando comparada com as abordagens alternativas, sendo facilmente replicável por investidores, instituições e participantes de mercado em geral.

PALAVRAS-CHAVE: seleção de carteiras; teoria nebulosa possibilística; carteira de variância mínima.

1 INTRODUCTION

The mean-variance methodology for the portfolio selection problem was established by Markowitz (Markowitz, 1952), which has been dominating the literature of modern finance since its appearance. It is assumed that all the investors treat the risk and asset returns as random variables. The expected risk and returns of a portfolio are referred to as the risk and investment return of the allocation, respectively. Thus, the model combines probability and optimization theories to model the behavior of economic agents under uncertainty. Over the last decades, mean-variance theory has played an important role in the development of modern portfolio selection theory, since requires a few number of parameters and the decision maker is able to produce relatively good results in their strategies (Sharpe, 1970; Merton, 1972; Elliot, Siu, & Badescu, 2010).

Traditional approaches of Markowitz's portfolio theory are of the crisp-stochastic category, which means that data and parameters are determined as crisp (real) numbers or unique distribution functions. Therefore, it is supposed that a decision maker is able to determine exactly the decision parameters and the unique input data, i.e., the distributions functions are known. However, in financial markets, the information is incomplete and complex, resulting to the impossibility to predict the future return and the actual risk of a portfolio precisely. In many situations, the input data are not precise but only fuzzy. Fuzzy theory is a powerful tool also used to describe the uncertain of financial environments where not only the financial markets but also the investment decision makers are subject to vagueness, ambiguity or fuzziness. Decision-making in a fuzzy environment was defined (Bellman & Zadeh, 1970) with a decision set which unifies a fuzzy objective and fuzzy constraint.

Since non-probabilistic factors affect the financial markets such that the return of risky asset is fuzzy uncertainty, a number of studies has been investigated for fuzzy portfolio selection problem (Watada, 1997; Leon, Lien, & Vercher, 2002). Fuzzy sets theory was used by (Bilbao-Terol, Gladish, Parra, & Uria, 2006) to the problem of portfolio selection using Sharpe's single-index as a model. In (Huang, 2007) was suggested an optimal portfolio selection model based on a new definition of risk for random fuzzy portfolio. Multi criteria decision making via fuzzy mathematical programming was applied by (Gupta, Mehlawat, & Saxena, 2008) to develop comprehensive models of asset portfolio optimization for the investors' with different levels of strategies, from conservative to more aggressive positions. In (Chen & Huang, 2009) was constructed a portfolio selection model that uses triangular fuzzy numbers to represent future return rates and future risks of equity mutual funds. In general, theses studies stated that when the returns are treated as fuzzy, the allocations provide better results in terms of risk and return than approaches that consider real (crisp) numbers, mainly in periods of high market changes.

As an extension of fuzzy sets theory, L. Zadeh (Zadeh, 1978) introduced possibilistic theory for dealing with incomplete information. In Zadeh's view, possibilistic distributions were meant to provide a graded semantics to natural language statements. Some studies have investigated possibilistic theory within the realm of fuzzy sets theory (Dubois & Prade, 1987; Carlsson & Fullér, 2001). In (Zhang & Nie, 2003) was proposed the notions of upper and lower possibilistic mean and possibilistic variances and covariances of fuzzy numbers as an extension of (Carlsson & Fullér, 2001). Due to this, several works have suggested the construction of efficient portfolios using possibilistic theory. Based on upper and lower possibilistic means and possibilistic variances, (Zhang, Wang, Chen, & Nie, 2007) proposed the notions of upper and lower possibilistic efficient portfolios. A possibilistic approach to select portfolios was suggested by (Carlsson, Fullér, & Majlender, 2002) with highest utility score under assumptions that the returns of assets are trapezoidal fuzzy numbers and short sales are not allowed on all risky assets. In (Zhang, Zhang, & Xiao, 2009) was dealt with the same problem but proposed a sequential minimal optimization algorithm to obtain the optimal portfolio.

Possibilistic portfolio adjusting problem with new added assets was studied by (Zhang, Xiao, & Xu, 2010), in order to fit changes in financial markets. They used a possibilistic portfolio adjusting model with transaction costs and bounded constraints on holding assets to show the case that investors do not need to invest total capital and to hold all assets in the portfolio for some required return levels. Also considering transaction costs, (Jana, Roy, & Mazimder, 2009) constructed a possibilistic model for portfolio selection. They quantify any potential return and risk by taking portfolio's liquidity into the objective function, which results in a multi-objective non-linear programming model for portfolio rebalancing. More recently, (Li, Zhang, & Xu, 2013) proposed a possibilistic portfolio model with Value-at-Risk constraint and risk-free based on the possibilistic mean and variance framework.

In this paper, we investigate the minimum variance possibility portfolio assuming that the expected rate of returns is a fuzzy number, represented by a Gaussian membership function. Some studies have suggested that the minimum variance portfolio, using real (crisp) data, provides higher adjusted returns to risk than other portfolios based on the classic mean-variance Markowitz's paradigm (Jagannathan & Ma, 2003; Jorion, 1991; Clark, Silva, & Thorley, 2006). The advantage of investing in the minimum variance portfolio is that it has a lower lever of risk than all the alternatives in the efficient frontier, and it is the only portfolio that investors do not have to provide a required level of return to find a position, which reduces the computational efforts in the respective optimization problem, mainly when large-scale portfolios are considered. As an application, the Brazilian equity market is considered in the numerical experiments. The minimum variance fuzzy possibilistic portfolio is compared to those of the following benchmarks: minimum variance portfolio constructed with real (crisp) numbers, the IBOVESPA equity index, an equally-weighted portfolio, and the maximum Sharpe ratio portfolio. Differently from the literature, in this work the possibilistic model is applied in a decision making portfolio selection process using actual data and compared with widely used benchmarks. Moreover, considering an emergent economy like Brazil, subject to market inefficiencies, this strategy can be easy replicable by individual and institutional investors alike, in order to show evidences to improve the liquidity in the equity market.

After this introduction, the remainder of the work proceeds as follows. Section 2 shows the basic concepts of the possibilistic mean and variance of a fuzzy number. The possibilistic portfolio model is presented in Section 3. Results and discussion are reported in Section 4. Finally, Section 5 concludes and suggests issues for further investigation.

2 POSSIBILISTIC MEAN AND VARIANCE

This section introduces some definitions for the construction of the fuzzy possibilistic portfolio selection model[1]. A fuzzy number X is a fuzzy set of the real line $\Re$ with a normal, fuzzy convex and continuous membership function of bounded support. Given a γ-level set of a fuzzy number X, $\gamma > 0$, according to (Carlsson & Fullér, 2001), the upper and lower possibilistic mean, with $[X]^\gamma = [x_1(\gamma), x_2(\gamma)]$, can be written, respectively, as:

$$M_U(X) = \frac{\int_0^1 pos[X \geq x_2(\gamma)] x_2(\gamma) d\gamma}{\int_0^1 pos[X \geq x_2(\gamma)] d\gamma}, \tag{1}$$

[1] For more details about possibilistic portfolio selection modeling see (Li et al., 2013) and (Zhang et al., 2007).

$$M_L(X) = \frac{\int_0^1 pos[X \leq x_1(\gamma)] x_1(\gamma) d\gamma}{\int_0^1 pos[X \leq x_1(\gamma)] d\gamma}, \quad (2)$$

where *pos* denotes possibility measure:

$$pos[X \geq x_2(\gamma)] = \sup_{t \geq x_2(\gamma)} X(t) = \gamma, \quad (3)$$

$$pos[X \leq x_1(\gamma)] = \sup_{t \leq x_1(\gamma)} X(t) = \gamma. \quad (4)$$

According to Eqs. (3) and (4), $M_U(X)$ and $M_L(X)$ become:

$$M_U(X) = 2\int_0^1 \gamma x_2(\gamma) d\gamma, \quad (5)$$

$$M_L(X) = 2\int_0^1 \gamma x_1(\gamma) d\gamma. \quad (6)$$

The possibilistic mean value of X can be written as the aritimetic mean of its lower and upper possibilistic mean values:

$$\bar{X} = \frac{M_U(X) + M_L(X)}{2}. \quad (7)$$

The upper and lower possibilistic variances and covariances of fuzzy numbers were introduced by (Zhang & Nie, 2003). The upper and lower possibilistic variances of a fuzzy number X with γ-level set $[X]^\gamma = [x_1(\gamma), x_2(\gamma)]$, $\gamma > 0$, are denoted, respectively, by:

$$\sigma_U^2 = 2\int_o^1 \gamma (M_U(X) - x_2(\gamma))^2 d\gamma, \quad (8)$$

$$\sigma_L^2 = 2\int_o^1 \gamma (M_L(X) - x_1(\gamma))^2 d\gamma. \quad (9)$$

Similarly, the possibilistic variance of a fuzzy number X can be written as:

$$\bar{\sigma}^2(X) = \frac{\sigma_U^2 + \sigma_L^2}{2}. \quad (10)$$

Alternatively, the upper and lower possibilistic covariances between fuzzy numbers X and Y, with γ-level set $[X]^\gamma = [x_1(\gamma), x_2(\gamma)]$ and $[Y]^\gamma = [y_1(y), y_2(y)]$, $\gamma \in [0,1]$, are represented as follows:

$$Cov_U(X,Y) = \frac{1}{2}\int_0^1 \gamma (M_U(X) - x_2(\gamma))(M_U(Y) - y_2(\gamma)) d\gamma, \quad (11)$$

$$Cov_L(X,Y) = \frac{1}{2}\int_0^1 \gamma (M_L(X) - x_1(\gamma))(M_L(Y) - y_1(\gamma)) d\gamma, \quad (12)$$

The possibilistic covariance between fuzzy numbers X and Y is:

$$\bar{Cov}(X,Y) = \frac{Cov_U(X,Y) + Cov_L(X,Y)}{2}. \quad (13)$$

Defined the possibilistic mean, variance and covariance, the next section presents the construction of the fuzzy possibilistic portfolio selection model, according to the mean-variance Markowitz's principle.

3 POSSIBILISTIC PORTFOLIO SELECTION MODEL

Let us considerer a universe A composed by n risky assets, and one risk-free available for investment, denoted by r_f. Let ε_i represents the return rate of asset i, $i = 1,\ldots,n$, which is a fuzzy number, and w_i be the proportion invested in asset i. The portfolio return, r_p, a fuzzy number, is computed by:

$$r_p = \sum_{i=1}^{n} w_i \varepsilon_i + r_f \left(1 - \sum_{i=1}^{n} w_i\right). \tag{14}$$

The possibilistic mean of the portfolio return r_p is given by:

$$\bar{M}(r_p) = \sum_{i=1}^{n} w_i \frac{M_U(\varepsilon_i) + M_L(\varepsilon_i)}{2} + r_f \left(1 - \sum_{i=1}^{n} w_i\right), \tag{15}$$

where $M_U(\varepsilon_i)$ and $M_L(\varepsilon_i)$ are the upper and lower possibilistic means of asset i return, respectively.

To compute the possibilistic variance of r_p, let us consider the following Lemma:
Lemma 1. *Let λ_1, $\lambda_2 \in \Re$ and let X and Y be fuzzy numbers, then*

$$\bar{\sigma}^2(\lambda_1 X + \lambda_2 Y) = \lambda_1^2 \bar{\sigma}^2(X) + \lambda_2^2 \bar{\sigma}^2(Y) + 2|\lambda_1 \lambda_2| \bar{Cov}\left(\phi(\lambda_1)X, \phi(\lambda_2)Y\right), \tag{16}$$

where $\phi(x)$ is a sign function of $x \in \Re$.

According to Lemma 1, the possibilistic variance of the portfolio return r_p can be written as:

$$\begin{aligned} \bar{\sigma}^2 &= \sum_{i=1}^{n} w_i^2 \bar{\sigma}_{\varepsilon_i}^2 + 2 \sum_{i>j=1}^{n} |w_i w_j| \bar{Cov}(\varepsilon_i, \varepsilon_j) \\ &= \sum_{i=1}^{n} w_i^2 \bar{\sigma}_{\varepsilon_i}^2 + 2 \sum_{i>j=1}^{n} w_i w_j \bar{Cov}(\varepsilon_i, \varepsilon_j). \end{aligned} \tag{17}$$

The possibilistic mean describes the portfolio return and the possibilistic variance represents the portfolio risk. Therefore, the possibilistic portfolio selection model can be formulated as follows:

$$\begin{aligned} & min\ \bar{\sigma}^2 = \sum_{i=1}^{n} w_i^2 \bar{\sigma}_{\varepsilon_i}^2 + 2 \sum_{i>j=1}^{n} w_i w_j \bar{Cov}(\varepsilon_i, \varepsilon_j) \\ & s.t.\ \sum_{i=1}^{n} w_i \frac{M_U(\varepsilon_i) + M_L(\varepsilon_i)}{2} + r_f \left(1 - \sum_{i=1}^{n} w_i\right) \geq \bar{r}, \\ & \sum_{i=1}^{n} w_i = 1, \\ & 0 \leq l_i \leq w_i \leq u_i \leq 1, i = 1,2,\ldots,n, \end{aligned} \tag{18}$$

where $\bar{r}$ is the underestimated expected return rate, l_i and u_i denote the lower bond and the upper bond on investment in asset i, respectively.

The set of all the possibilistic efficient portfolios comprises the possibilistic efficient frontier, which can be traced out by solving the portfolio problem for all possible values of $\bar{r}$. In this work, our focus is on the minimum variance portfolio, i.e., with the minimum level of risk, independent of the expected return rate $\bar{r}$. Moreover, without loss of generality, we do not imposed as a constraint, defined bonds to the proportion of capital invested on each asset, w_i. Thus, the minimum variance possibilistic portfolio is given by solving the following problem:

$$\begin{aligned} & min\ \bar{\sigma}^2 = \sum_{i=1}^{n} w_i^2 \bar{\sigma}_{\varepsilon_i}^2 + 2 \sum_{i>j=1}^{n} w_i w_j \bar{Cov}(\varepsilon_i, \varepsilon_j) \\ & s.t. \sum_{i=1}^{n} w_i = 1, \\ & 0 \leq w_i \leq 1, i = 1, 2, \ldots, n. \end{aligned} \tag{19}$$

The next step is to define the fuzzy variables, i.e., the portfolio rate of return. Suppose that the return rate of asset i is a Gaussian fuzzy variable, i.e., $\varepsilon_i \sim G(\mu_i, \sigma_i)$, and its membership function and level set are, respectively:

$$X_{\varepsilon_i}(t) = \exp\left(\frac{t-\mu_i}{\sigma_i}\right)^2, \tag{20}$$

$$[\varepsilon_i]^\gamma = \left[\mu_i - \sigma_i\sqrt{ln\gamma^{-1}}, \mu_i + \sigma_i\sqrt{ln\gamma^{-1}}\right], \tag{21}$$

with $\gamma \in (0,1), i = 1, 2, \ldots, n$.

In order to construct the portfolio fuzzy distribution, let us consider the following Lemma:

Lemma 2. *Let λ_1, $\lambda_2 \in \Re$ and let X and Y be fuzzy numbers, then*

$$\bar{M}(\lambda_1 X + \lambda_2 Y) = \lambda_1 \bar{M}(X) + \lambda_2 \bar{M}(Y). \tag{22}$$

According to Lemma 2, the possibilistic mean value of $\sum_{i=1}^{n} w_i \varepsilon_i$ is calculated as:

$$\bar{M}\left(\sum_{i=1}^{n} w_i \varepsilon_i\right) = \sum_{i=1}^{n} w_i \bar{M}(\varepsilon_i) = \sum_{i=1}^{n} w_i \mu_i. \tag{23}$$

From Eqs. (5) and (6), the upper and lower possibilistic means of ε_i can be calculated, respectively, by:

$$\begin{aligned} M_U(\varepsilon_i) &= 2\int_0^1 \gamma\left(\mu_i + \sigma_i\sqrt{ln\gamma^{-1}}\right) d\gamma \\ &= \mu_i + 2\sigma_i \int_0^1 \gamma\sqrt{ln\gamma^{-1}} d\gamma \\ &= \mu_i + \sigma_i \frac{\sqrt{\pi}}{2\sqrt{2}}, \end{aligned} \tag{24}$$

$$\begin{aligned} M_L(\varepsilon_i) &= 2\int_0^1 \gamma\left(\mu_i - \sigma_i\sqrt{ln\gamma^{-1}}\right) d\gamma \\ &= \mu_i - \sigma_i \frac{\sqrt{\pi}}{2\sqrt{2}}. \end{aligned} \tag{25}$$

In this way, we can obtain the upper and lower possibilistic portfolio variances:

$$\begin{aligned}
\sigma_U^2 &= 2\int_0^1 \gamma[M_U(\varepsilon_i) - x_2(\gamma)]^2 d\gamma \\
&= 2\int_0^1 \gamma\left(\mu_i + \sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}} - \mu_i - \sigma_i\sqrt{ln\gamma^{-1}}\right)^2 d\gamma \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i^2, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\sigma_L^2 &= 2\int_0^1 \gamma[M_L(\varepsilon_i) - x_1(\gamma)]^2 d\gamma \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i^2. \qquad (27)
\end{aligned}$$

The fuzzy possibilistic portfolio variance can be written as:

$$\bar{\sigma}_{\varepsilon_i}^2 = \frac{\sigma_U^2 + \sigma_L^2}{2} = \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i^2. \qquad (28)$$

Considering Eqs. (11) and (12), the upper and lower possibilistic covariances can be calculated by:

$$\begin{aligned}
Cov_U(\varepsilon_i, \varepsilon_j) &= 2\int_0^1 \gamma(M_U(\varepsilon_i) - x_2(\gamma))(M_U(\varepsilon_j) - y_2(\gamma))d\gamma \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
Cov_L(\varepsilon_i, \varepsilon_j) &= 2\int_0^1 \gamma(M_L(\varepsilon_i) - x_1(\gamma))(M_L(\varepsilon_j) - y_1(\gamma))d\gamma \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j. \qquad (30)
\end{aligned}$$

The possibilistic covariance is given by:

$$\begin{aligned}
\bar{Cov}(\varepsilon_i, \varepsilon_j) &= \frac{Cov_U(\varepsilon_i, \varepsilon_j) + Cov_L(\varepsilon_i, \varepsilon_j)}{2} \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j. \qquad (31)
\end{aligned}$$

Taking into account Lemma 1 and considering $w_i \geq 0$, the possibilistic variance of the portfolio return, $\sum_{i=1}^n w_i\varepsilon_i$, is:

$$\begin{aligned}
\bar{\sigma}^2_{\sum_{i=1}^n w_i\varepsilon_i} &= \sum_{i=1}^n w_i^2\bar{\sigma}_{\varepsilon_i}^2 + 2\sum_{i>j=1}^n w_i w_j \bar{Cov}(\varepsilon_i, \varepsilon_j) \\
&= \sum_{i=1}^n \left(\frac{1}{2} - \frac{\pi}{8}\right) w_i^2\sigma_i^2 + 2\sum_{i>j=1}^n \left(\frac{1}{2} - \frac{\pi}{8}\right) w_i w_j \sigma_i\sigma_j \\
&= \left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n w_i\sigma_i\right)^2. \qquad (32)
\end{aligned}$$

Then, the possibilistic fuzzy portfolio return $\sum_{i=1}^{n} w_i\varepsilon_i$ is Gaussian distributed with the following parameters:

$$\sum_{i=1}^{n} w_i\varepsilon_i \sim G\left(\sum_{i=1}^{n} w_i\mu_i, \left(\frac{1}{2}-\frac{\pi}{8}\right)\left(\sum_{i=1}^{n} w_i\sigma_i\right)^2\right), \tag{33}$$

and membership function:

$$X_{r_p}(t) = \exp\left\{\frac{-(t-\sum_{i=1}^{n} w_i\mu_i)^2}{\left(\frac{1}{2}-\frac{\pi}{8}\right)(\sum_{i=1}^{n} w_i\sigma_i)^2}\right\}. \tag{34}$$

From Eqs. (14), (24) and (25), the upper and lower possibilistic means of the portfolio return, r_p, are:

$$M_U(r_p) = \sum_{i=1}^{n}\left(\mu_i + \sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}} - r_f\right)w_i + r_f, \tag{35}$$

$$M_L(r_p) = \sum_{i=1}^{n}\left(\mu_i - \sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}} - r_f\right)w_i + r_f. \tag{36}$$

The possibilistic mean of r_p is given by:

$$\bar{M} = \frac{M_U(r_p)+M_L(r_p)}{2} = \sum_{i=1}^{n}(\mu_i - r_f) + r_f. \tag{37}$$

Then, in the case of the asset returns are fuzzy Gaussian distributed, the possibilistic portfolio selection model in (18) can be reformulated as:

$$\begin{aligned}
&min\ \bar{\sigma}^2 = \left(\frac{1}{2}-\frac{\pi}{8}\right)\left(\sum_{i=1}^{n} w_i^2\sigma_i^2 + 2\sum_{i>j=1}^{n} w_iw_j\sigma_i\sigma_j\right)\\
&s.t.\ \sum_{i=1}^{n} w_i(\mu_i - r_f) + r_f \geq \bar{r},\\
&\sum_{i=1}^{n} w_i = 1,\\
&0 \leq l_i \leq w_i \leq u_i \leq 1, i = 1,2,\ldots,n.
\end{aligned} \tag{38}$$

Similarly, the minimum variance fuzzy possibilistic portfolio, as in (19), with no restriction to the investment proportions w_i, is given by:

$$\begin{aligned}
&min\ \bar{\sigma}^2 = \left(\frac{1}{2}-\frac{\pi}{8}\right)\left(\sum_{i=1}^{n} w_i^2\sigma_i^2 + 2\sum_{i>j=1}^{n} w_iw_j\sigma_i\sigma_j\right)\\
&s.t. \sum_{i=1}^{n} w_i = 1,\\
&0 \leq w_i \leq 1, i = 1,2,\ldots,n.
\end{aligned} \tag{39}$$

4 COMPUTATIONAL EXPERIMENTS

This paper evaluates the performance of minimum variance fuzzy possibilistic portfolio in the Brazilian equity market in the case that assets return is a Gaussian fuzzy variable. Its results are compared to those of the following benchmarks: minimum variance portfolio estimated by real (crisp) numbers, the IBOVESPA equity index, an equally-weighted portfolio, and the maximum Sharpe ratio portfolio. The IBOVESPA is the most used index for the Brazilian equity market, which includes the most traded assets in the Brazilian market. Besides there are no reasons of IBOVESPA be an efficient portfolio, it is frequently used as market portfolio in finance studies, and translates the Brazilian equity market performance. As follows we describe the data, the benchmark models, the results and its discussion.

4.1 Data

The data comprises daily closing prices of all assets traded in the Brazilian equity market from January 2000 to December 2012. Assets from the same corporations were also considered, e.g., preferred and ordinary shares. We consider only the assets with a positive negotiation during the period considered, which reduces the database in a number of 314 distinct assets. Moreover, daily series from IBOVESPA index were also collected. The risk free investment was represented by the CDI rate. CDI, or Interbank Deposit Certificate, is the indicator computed by the average of the interbank operations rates, widely used as risk free interest rate in the Brazilian financial markets. It is provided in annual basis. Thus, the CDI daily returns, CDI_{daily}, were calculated according to:

$$\text{CDI}_{daily} = (1 + \text{CDI}_{annual})^{\frac{1}{252}} - 1, \quad (40)$$

where CDI_{annual} is the CDI value, in annual basis.

4.2 Benchmarks

One of the benchmarks considered in this paper is the minimum variance portfolio using real (crisp) numbers. Let be n a universe of available assets and a portfolio represented by weights $\mathbf{w} = [w_1, w_2, ..., w_n]^T$, the minimum variance portfolio using crisp data, denoted by MVP_C, is obtained by the solution of the following problem:

$$\min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}$$

$$s.t. \sum_{i=1}^{n} w_i = 1, 0 \leq w_i \leq 1 \; \forall \, i, \quad (41)$$

where Σ is the assets covariance matrix, estimated by the sample covariance matrix[2].

The sample covariance matrix supposes the hypothesis that the returns are i.i.d., and is computed using a sample from returns time series. The covariance between assets i and j is estimated by:

$$\hat{\sigma}_{ij} = \frac{1}{T} \sum_{i=1}^{n} (r_{i,t} - \bar{r}_i)(r_{j,t} - \bar{r}_j), \quad (42)$$

[2] (Jagannathan & Ma, 2003) stated that the use of the sample covariance matrix provides results as good as from more sophisticated and robust estimators.

where $r_{i,t}$ is the return from asset i at t, $\bar{r}$ is the sample average of assets returns, and T is the sample size.

Alternatively, this paper also used as benchmark the minimum variance portfolio that maximizes the Sharpe ratio, MVP$_S$. The Sharpe ratio of a portfolio p is defined by $SR = (\bar{r}_p - r_f)/\sigma_p$, where $\bar{r}_p$ is the portfolio average return, r_f is the risk free interest rate and σ_p is the portfolio volatility. The MVP$_S$ is the solution of the following optimization problem:

$$\max_{\mathbf{w}} \frac{\mathbf{w}^T \bar{\mathbf{r}}_p}{\sqrt{\mathbf{w}^T \Sigma \mathbf{w}}}$$

$$s.t. \sum_{i=1}^{n} w_i = 1, 0 \leq w_i \leq 1, \; \forall\, i, \tag{43}$$

where $\bar{\mathbf{r}}_p = [\bar{r}_1, \bar{r}_2, ..., \bar{r}_n]^T$ is the vector of average returns.

The equally-weighted portfolio, EWP, has equal weights for all its assets. If we have n assets available, the weights that define the EWP are given by:

$$w_i = \frac{1}{n}, \; \forall\, i. \tag{44}$$

One must note that all of these previous benchmarks were computed using real (crisp) numbers. Finally, the minimum variance fuzzy possibilistic portfolio in (39), MVP$_{FP}$, is also compared with the IBOVESPA equity market index. For the fuzzy possibilistic model each asset return is fuzzy Gaussian distributed, i.e., $\varepsilon_i \sim G(\mu_i, \sigma_i)$. The parameters μ_i and σ_i were estimated according to the sample mean and sample standard deviation from the historic data, respectively.

4.3 Performance assignment

The performance of the minimum variance portfolio models was measured in terms of annualised return (A$_R$), cumulative return (C$_R$), annualized volatility (A$_V$) and maximum loss or maximum drawdown (ML), defined as follows, respectively:

$$\mathrm{A}_R = 252 \cdot \frac{1}{T} \sum_{t=1}^{T} r_{p,t}, \tag{45}$$

$$\mathrm{C}_R = \sum_{t=1}^{T} r_{p,t}, \tag{46}$$

$$\mathrm{A}_V = \sqrt{252} \cdot \sqrt{\frac{1}{T-1} \sum_{t=1}^{T} (r_{p,t} - \bar{r}_p)^2}, \tag{47}$$

$$\mathrm{ML} = \min_{l=1,...,t; t=1,...,T} \sum_{l=1}^{t} r_{p,l}, \tag{48}$$

where r_p and $\bar{r}_p$ represent the return and the average return of a portfolio, respectively.

Moreover, we consider the Sharp ratio (SR) of the portfolios:

$$\mathrm{SR} = \frac{r_p - r_f}{\sigma_p}, \tag{49}$$

where σ_p indicates the risk of the portfolio, measured by the portfolio returns standard deviation. It is a commonly used measure to compare investments which takes into account the trade-off between risk and return.

Finally, we compute the systemic risk of each portfolio by estimating the coefficient beta (β) of the following regression:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i(r_{IBOV,t} - r_{f,t}) + \varepsilon_{i,t}, \tag{50}$$

where $r_{i,t}$ is the return of portfolio i at t, $t = 1, 2, ..., T$, α and β are the parameters, r_{IBOV} is the return of IBOVESPA equity market index, and $\varepsilon \sim N(0,1)$.

4.4 Results

Portfolios were composed by the models using past data from January 2000 through December 2007. Then, their performance were evaluated for the remaining period, i.e., from January 2008 to December 2012. Table 1 summarizes the results of the minimum variance portfolios in terms of annualized return, cumulative return, annualized volatility, Sharpe ratio, maximum loss and correlation to IBOVESPA. Moreover, portfolio's systemic risk was measured by the coefficient β, estimated from Eq. (50).

Tab. 1: Minimum Variance Portfolios Performance

Metrics	MVP_{FP}	MVP_C	MVP_S	EWP	IBOV
A_R	35.17%	29.55%	26.33%	20.64%	17.34%
C_R	147.01%	113.45%	101.87%	99.14%	87.20%
V_A	17.03%	22.34%	25.70%	27.20%	34.66%
SR	0.70	0.67	0.58	0.16	0.19
ML	-74.88%	-86.19%	-84.01%	-86.91%	-87.49%
$IBOV_{corr}$	0.65	0.69	0.70	0.84	-
beta (β)	0.41	0.51	0.55	0.67	-

In terms of annualized return, the minimum variance fuzzy possibilistic portfolio provided better results than all remaining portfolios, twice higher than the main benchmark in the Brazilian equity market, the IBOVESPA, which showed the lowest annualized return. The same results were found considering the cumulative return metric. Minimum variance portfolio estimated using real number is the second best model by its annualized and cumulative returns.

MVP_S, EWP and IBOVESPA are the riskiest portfolios since showed the highest values of annualized volatility. Therefore, in these terms, the fuzzy possibilistic portfolio provides the better combination of return and risk, since it is the more conservative approach (lowest level of annualized volatility). The Sharpe ratio indicates the portfolio risk premium by each unit of risk and, according with the results (Table 1), the Sharpe ratios of MVP_{FP}, MVP_C and MVP_S portfolios are statistically superior to the IBOVESPA ratio, considering the statistical test for Sharpe ratio comparisons of (Ledoit & Wolf, 2008) with a significance level of 1%.

The maximum loss, or maximum drawdown, which is a risk indicator widely used by portfolio managers, shows similar results for all approaches with an approximate average of -83.90%. This several loss is due to the recent Subprime crisis, that started in the US economy and affected other markets, including emergent economies like Brazil. All portfolios present high correlation with the IBOVESPA index. EWP and MVP_{FP} have the highest and lowest correlations to IBOVESPA, respectively (Table 1). It means that the portfolios move in the same direction of the equity market index, besides their lower levels of risk, confirming the study of (Clark et al., 2006), which stated that minimum variance portfolios are composed by assets with lower risk than the market portfolio.

According to the coefficient beta (β), which is a measure of systemic risk, one may see that the MVP_{FP} performs better than the other approaches, with a lower systemic risk, i.e., for a variation of 1% in the market portfolio, the fuzzy possibilistic portfolio varies 0.41%. The fact that the beta coefficient does not explain the portfolios returns can be a reflect of some other risk factors that are not priced in the model, since the relation of return and risk is not verified empirically.

Finally, Table 2 shows the number of assets allocated in each portfolio and indicates that the minimum variance portfolios require a few number of assets, being easy replicable, except the equally-weighted portfolio.

Tab. 2: Portfolio's Number of Assets

Portfolios	MVP_{FP}	MVP_C	MVP_S	EWP
# assets	17	22	43	314

Summarizing, the results showed that the minimum variance fuzzy possibilistic portfolio model provides better results than all alternative techniques with a few number of assets, indicating the high adequacy in uncertain environments like the Brazilian equity market.

5 CONCLUSION

Portfolio selection is one of the most challenging problem in finance. The mean-variance methodology is widely used by market participants in order to construct their portfolios, since it is a well-known technique that a decision maker should determine a few number of parameters and can produce considerable results. The key issue of the conventional probabilistic mean-variance approach is to use the expected return of a portfolio as the investment return and to use the variance (or standard deviation) of the expected returns of the portfolio as the investment risk, under the assumption that the future assets behavior can be reflected by data in the past. However, because of the information incompleteness and the complexity of the financial markets, it is impossible to precisely predict the future return and the actual risk of a portfolio, since these variables are fuzzy uncertainty.

This paper evaluates the performance of minimum variance fuzzy possibilistic portfolio in the Brazilian equity market. In the efficient frontier of investments, the minimum variance portfolio represents a more conservative approach, i.e., the one with the lower level of risk. In the possibilistic model, we assume that the portfolio return is fuzzy, represented by a Gaussian membership function, which models a scenario subject to fuzziness since non-probabilistic factors affect the financial markets. The suggested methodology was compared with the following methods: a minimum variance portfolio estimated with real (crisp) data, the IBOVESPA index which is used as the main benchmark in the Brazilian equity market, an equally-weighted portfolio, and the maximum Sharpe ratio portfolio.

The results show that the minimum variance fuzzy possibilistic portfolio has higher returns with a lower level of risk compared to all alternative approaches. Moreover, its beta coefficient, as a measure of systemic risk, is lower than one, which indicates that the fuzzy possibilistic portfolio has lower risk than the market portfolio, i.e., the IBOVESPA index. It indicates that this strategy provides good results in a simple procedure with a few number of assets, being easy replicable by investors, institutions and all market participants in general. Future works shall include transaction costs in the portfolio model, short positions, as well as evaluate its performance with rebalancing, in order to capture more adequately the market fluctuations.

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